Dynamic stresses at a moving crack tip in a model of fracture propagation

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The steady-state propagation of a mode I crack in a laterally strained strip is examined. The displacement field satisfies the usual equation of motion for an isotropic two-dimensional elastic medium while the tractions on the fracture surface include a viscous dissipative term recently introduced by J. S. Langer and H. Nakanishi [Phys. Rev. E **48**, 439 (1993)]. The stress field at the crack tip is calculated and found to change qualitatively as the crack speed increases beyond a certain critical value. Such a dynamical modification of the crack tip stress field has an interesting indication that steady-state crack propagation above the critical speed may be unstable.

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I. INTRODUCTION

There is a fundamental unresolved puzzle in dynamic fracture [1]. The speed of a moving crack is commonly believed to be limited only by the rate at which stored elastic energy is transported to the crack tip. The limiting speed should, therefore, be the Rayleigh wave speed c_R which is the velocity at which elastic waves travel along the free fracture surface. However, experimentally obtained maximum crack speeds are much lower, being typically around half of the Rayleigh wave speed [2]. Recent measurements by Fineberg *et al.* [3] reveal a dynamic instability in the fracture of a brittle plastic. The crack tip oscillates and the fracture surface roughens when the velocity is larger than about $0.34c_R$. The onset of a dynamic instability may explain why observed crack speeds are so much lower than the theoretical maximum.

The possible existence of an instability in crack propagation was first suggested by a calculation of Yoffe [4]. She considered a steadily propagating crack in a plate of isotropic elastic material. The crack has the form of a straight narrow slit and maintains its length while moving. The angular variation of the circumferential tensile stress acting across a small fixed distance from the head of the crack was calculated. She found that for small crack speeds, this stress component has a maximum along the direction of crack growth. For crack speeds greater than about 60% of the transverse wave speed c_t of the material, however, the maximum shifts to a direction inclined at about 60° to that of crack growth. Yoffe then suggested if a crack is to advance in a direction normal to the maximum direct stress, this feature could account for the tendency of rapidly growing cracks to curve and bifurcate into several branched cracks in brittle materials. This could also explain why steady-state crack propagation cannot occur at high speeds. However, for a two-dimensional elastic plate, c_R is about $0.87-0.93$ of c_t (depending on the value of Poisson's ratio); the critical speed that she found is, therefore, higher than the maximum cracks speeds observed in experiments and particularly higher than the critical velocity observed by Fineberg *et al.* [3].

The problem set up by Yoffe is not physically realistic. Craggs [5] repeated the calculation for a semi-infinite crack and Baker [6] considered a transient crack growth problem. Both found the same asymptotic stress field close to the crack tip. It is now well known that this asymptotic stress field is universal and has a square root singularity at the crack tip. Because of this singularity, only the angular distribution at a small fixed distance from the crack tip can be studied, as was done by Yoffe. However, no real material can support such a singular stress distribution. The potentially large stresses in the vicinity ahead of the crack are relieved through plastic flow or some other inelastic processes. If this inelastic crack tip region is small, compared to the crack length and the body dimensions, the stress distribution outside the inelastic zone is adequately described by Yoffe's singular stress field. However, one would expect that the stresses right at the crack tip are more relevant in governing crack advance. Thus, one should study the principal stress components at the tip. Conventionally, the inelastic crack tip region is modeled by the so-called cohesive zone [7]. The opening of the crack is resisted by a cohesive stress which acts in a small region just behind the crack tip. The Barenblatt condition is applied to make the stress nonsingular which also determines the length of the cohesive zone.

In Yoffe's work, the crack speed is just taken to be a constant. The existence of such steady-state solutions cannot be demonstrated within the framework of her calculation. In order for steady-state crack propagation to be physically realistic, some velocity-dependent dissipation mechanisms have to be present. Langer [8] has studied models of crack propagation with different dissipation mechanisms, such as velocity-dependent friction and Kelvin viscoelasticity. In these models, the dissipative mechanism acts in the bulk of the material, i.e., it appears as a term in the equation of motion for the displacement field. The crack tip stress field is regularized by the cohesive-zone concept described above. More recently, Langer

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and Nakanishi [9] have introduced a new model of crack propagation. The new ingredient is a novel viscous dissipation which acts only on the fracture surface. The stress field at the crack tip was found to be nonsingular even without applying the usual Barenblatt condition. Furthermore, various interesting consequences were found [10].

This paper will report the investigation of using this new element of viscous dissipation to study steady-state propagation of a mode I crack. The crack moves along the centerline of an infinite strip which has a finite width and is laterally strained. The crack tip stress field is calculated and it is found that the two principal stresses act on directions along and perpendicular to the centerline. For small crack speeds, the principal stress acting normal to the direction of crack growth (the centerline) is larger, as one would expect. However, when the crack moves faster than some critical speed, the principal stress acting along the centerline becomes larger. When this happens, the crack will grow in a direction parallel to the maximum principal stress. Moreover, the shear stress along any line that makes a small angle with the centerline will then be acting in a direction such that any small deviation of the crack growth from the original one is enhanced. These features indicate that steady-state crack propagation at speeds faster than the critical value may be unstable.

II. SPECIFICS OF THE PROBLEM

Consider an infinite strip of elastic material with half-width W as shown in Fig. 1. The two edges are clamped rigidly and displaced by an outward distance Δ. A mode I crack travels from right to left along the x axis. The displacement field of the material, $\mathbf{u}(x,y,t)$, satisfies the usual two-dimensional elastic wave equation:

$$\ddot{\mathbf{u}}=c_t^2\nabla^2\mathbf{u}+(c_l^2-c_t^2)\nabla(\nabla\cdot\mathbf{u}) , \tag{2.1}$$

with the boundary conditions

$$u_x(x,\pm W,t)=0 , \quad -\infty<x<\infty \tag{2.2}$$

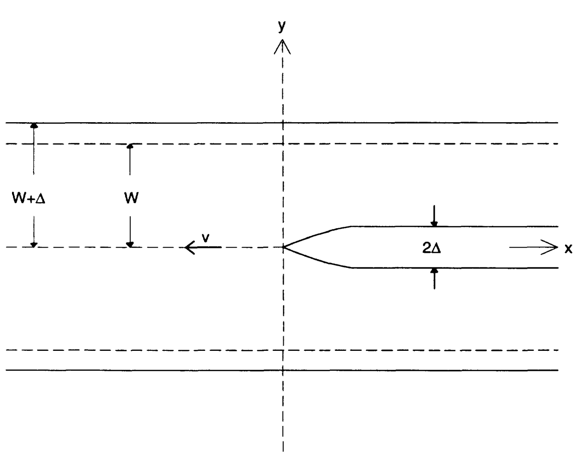

FIG. 1. Schematic diagram of a mode I crack propagating at speed v along the centerline of a laterally strained strip of unstrained half-width W.

and

$$u_y(x,\pm W,t)=\pm\Delta , \quad -\infty<x<\infty . \tag{2.3}$$

The transverse and longitudinal wave speeds c_t and c_l are given by

$$c_t=\left[\frac{E}{2\rho(1+\nu)}\right]^{1/2} , \quad c_l=\sqrt{\kappa}c_t ,$$

with

$$\kappa=\begin{cases}\dfrac{2(1-\nu)}{1-2\nu} & \text{(plane strain)}\\[2mm] \dfrac{2}{1-\nu} & \text{(plane stress)} ,\end{cases} \tag{2.4}$$

where E and ν are, respectively, Young's modulus and Poisson's ratio and ρ is the mass density of the material. Be definition, $u_y(x,0)=0$ for the unbroken part of the strip. The stress $\sigma_{\alpha\beta}$ (α and β are two-dimensional coordinate indices) is related to the displacement by standard elasticity theory:

$$\sigma_{\alpha\beta}=\frac{E}{1+\nu}\left[\left[\frac{\kappa}{2}-1\right](\nabla\cdot\mathbf{u})\delta_{\alpha\beta}+\tfrac{1}{2}(\partial_\alpha u_\beta+\partial_\beta u_\alpha)\right] . \tag{2.5}$$

The condition

$$\sigma_{xy}(x,0,t)=0 , \quad -\infty<x<\infty \tag{2.6}$$

assures symmetry about the x axis.

To define the problem completely, one needs to specify the stress acting on the fracture surface. As mentioned in the Introduction, we follow Ref. 7, hereafter referred to as LN, and assume

$$\sigma_{yy}(x,0,t)=\begin{cases}\sigma_0(x,t) , & u_y(x,0,t)=0\\[2mm] \sigma_c\{u_y\}-\eta\left.\dfrac{\partial^2\dot{u}_y}{\partial x^2}\right|_{y=0} , & u_y(x,0,t)>0 .\end{cases} \tag{2.7}$$

On the right hand side of Eq. (2.7), $\sigma_0(x)$ is the as yet undetermined stress acting across the fracture line ahead of the crack tip, where $u_y=0$. Behind the crack tip where $u_y>0$, the first term $\sigma_c\{u_y(x,0,t)\}$ is the cohesive stress acting between the open crack faces and has the form

$$\sigma_c\{u_y\}=\begin{cases}\bar{\sigma} , & 0<u_y<\delta\\ 0 , & u_y>\delta ,\end{cases} \tag{2.8}$$

where $\bar{\sigma}$ is the yield stress and δ is the range of the cohesive force. The second term is a viscous damping stress acting on the fracture surface with η being a phenomenological parameter. As discussed in LN, such a term is not forbidden by any conservation law or symmetry principle and is a simple way to introduce a localized dissipative mechanism by adding only one new length scale.

It is convenient to express all lengths in units of the half-width W and all velocities in units of the transverse wave speed c_t. Thus, (2.1) becomes

$$\ddot{\mathbf{u}}=\nabla^2\mathbf{u}+(\kappa-1)\nabla(\nabla\cdot\mathbf{u}) , \tag{2.9}$$

and the boundary conditions are

$$u_x(x,\pm1,t)=0 , \quad -\infty<x<\infty \tag{2.10}$$

and

$$u_y(x,\pm1,t)=\pm\epsilon_\infty , \quad -\infty<x<\infty , \tag{2.11}$$

where $\epsilon_\infty=\Delta/W$ is the externally applied strain. Equation (2.7) becomes

$$\left[1-\frac{2}{\kappa}\right]\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}\Bigg|_{y=0} = \begin{cases}\epsilon_0(x,t) , & u_y(x,0,t)=0\\ \epsilon_c\{u_y\}-\tilde{\eta}\dfrac{\partial^2\dot{u}_y}{\partial x^2}\Bigg|_{y=0} , & u_y(x,0,t)>0 .\end{cases} \tag{2.12}$$

Here, $\epsilon_0(x,t)=\sigma_0(x,t)/\mu$, $\epsilon_c\{u_y\}=\sigma_c\{u_y\}/\mu$, and $\tilde{\eta}=\eta c_t/\mu W^2$, with $\mu=E\kappa/2(1+\nu)$.

III. WIENER-HOPF SOLUTIONS

To solve (2.9), we write [11] the vector displacement field $\mathbf{u}$ in terms of two scalar potentials:

$$\mathbf{u}=\nabla\phi+\nabla\times\psi\hat{\mathbf{z}} , \tag{3.1}$$

which satisfy the following equations:

$$\nabla^2[\ddot{\phi}-\kappa\nabla^2\phi]=0 , \tag{3.2}$$

$$\nabla^2[\ddot{\psi}-\nabla^2\psi]=0 . \tag{3.3}$$

We look for steady-state solutions in a frame of reference moving with speed $-v$ and write $\phi(x,y,t)=\tilde{\phi}(\xi,y)$ with $\xi=x+vt$ (similarly for ψ and other quantities). The tip of the crack is at $\xi=0$; thus, $\tilde{u}_y(\xi,0)\equiv U(\xi)=0$ for $\xi<0$.

The Wiener-Hopf method for solving this problem starts with the Fourier representation of $\tilde{\phi}$ and $\tilde{\psi}$:

$$\tilde{\phi}(\xi,y)=\frac{\epsilon_\infty y^2}{2}+\int\frac{dk}{2\pi}e^{ik\xi}\hat{\phi}(k,y) , \tag{3.4}$$

$$\tilde{\psi}(\xi,y)=\int\frac{dk}{2\pi}e^{ik\xi}\hat{\psi}(k,y) . \tag{3.5}$$

The first term in (3.4) produces a displacement field that satisfies the boundary conditions (2.10) and (2.11) and is the uniformly strained state of the strip in the absence of a crack. The most general forms for $\hat{\phi}$ and $\hat{\psi}$ which satisfy (3.2) and (3.3) and produce a displacement field that satisfies (2.9) are

$$\hat{\phi}(k,y)=a_1(k)\cosh(k\beta_l y)+a_2(k)\sinh(k\beta_l y) +b_1(k)\cosh(ky)+b_2(k)\sinh(ky) , \tag{3.6}$$

$$\hat{\psi}(k,y)=a_3(k)\cosh(k\beta_t y)+a_4(k)\sinh(k\beta_t y) -ib_2(k)\cosh(ky)-ib_1(k)\sinh(ky) , \tag{3.7}$$

where $\beta_l^2=1-v^2/\kappa$ and $\beta_t^2=1-v^2$. However, it turns out that the terms with coefficients b_1 and b_2 drop out from the expressions for u_x and u_y. Since we are actually interested in the displacement field $\mathbf{u}$ and not the scalar potentials, we can take

$$\hat{\phi}(k,y)=a_1(k)\cosh(k\beta_l y)+a_2(k)\sinh(k\beta_l y) , \tag{3.8}$$

$$\hat{\psi}(k,y)=a_3(k)\cosh(k\beta_t y)+a_4(k)\sinh(k\beta_t y) . \tag{3.9}$$

Because of the symmetry of the problem, we need to consider only the upper-half of the strip $0\le y\le1$. The three boundary conditions (2.6), (2.10), and (2.11) are used to determine the four coefficients a_i's in terms of one of them and we get

$$a_1(k)=\frac{i(1+\beta_t^2)}{2\beta_l}a_3(k) , \tag{3.10}$$

$$a_2(k)=iG(k)a_3(k) , \tag{3.11}$$

$$a_4(k)=H(k)a_3(k) , \tag{3.12}$$

where

$$G(k)=\frac{1}{D(k)}\left\{\beta_t-\frac{(1+\beta_t^2)}{2\beta_l}[\beta_t\beta_l\cosh(k\beta_t)\cosh(k\beta_l)-\sinh(k\beta_l)\sinh(k\beta_t)]\right\} , \tag{3.13}$$

$$H(k)=\frac{1}{D(k)}\left[\cosh(k\beta_l)\cosh(k\beta_t)-\frac{(1+\beta_t^2)}{2}-\beta_t\beta_l\sinh(k\beta_l)\sinh(k\beta_t)\right] , \tag{3.14}$$

and

$$D(k)=\beta_t\beta_l\sinh(k\beta_l)\cosh(k\beta_t)-\cosh(k\beta_l)\sinh(k\beta_t) . \tag{3.15}$$

The fourth equation on the a_i's is obtained by the Fourier transform of (2.12):

$$-\hat{P}(k)ik\hat{U}^{(+)}(k)=ik\hat{\epsilon}_0^{(-)}(k)+ik\hat{\epsilon}_c^{(-)}(k)-\epsilon_\infty , \tag{3.16}$$

where

$$\hat{P}(k)=\hat{F}(k)+i\tilde{\eta}vk^3 , \tag{3.17}$$

$$\hat{F}(k)=\frac{2k}{\kappa v^2}[(1+\beta_t^2)G(k)-2\beta_t H(k)] , \tag{3.18}$$

$$\hat{\epsilon}_0^{(-)}(k)=\int_{-\infty}^{0} d\xi\, e^{-ik\xi}[\tilde{\epsilon}_0(\xi)-\epsilon_\infty] , \tag{3.19}$$

$$\hat{\epsilon}_c^{(+)}(k)=\int_0^{\infty} d\xi\, e^{-ik\xi}\epsilon_c\{U(\xi)\} . \tag{3.20}$$

Here, $\hat{U}^{(+)}(k)$ is the Fourier transform of $U(\xi)$ and is related to the coefficients a_i's as follows:

$$\hat{U}^{(+)}(k)=k\beta_l a_2(k)-ika_3(k) . \tag{3.21}$$

The superscript $(+)$ in, e.g., $\hat{U}^{(+)}(k)$, indicates that the function has singularities only in the upper-half k plane. This is so as it is the Fourier transform of $U(\xi)$ which is nonzero only for positive ξ. The superscript $(-)$ has the corresponding opposite meaning. It is checked that $\hat{F}(k)$ reduces to the Knauss integral kernel for the case of a stationary crack [12], as $v\to 0$.

The three distinguished components of the stress tensor can then be written as

$$\tilde{\sigma}_{xx}(\xi,y)=\mu\left[1-\frac{2}{\kappa}\right]\epsilon_\infty+\int\frac{dk}{2\pi}e^{ik\xi}K_{xx}(k,y)\hat{U}^{(+)}(k) , \tag{3.22}$$

$$\tilde{\sigma}_{yy}(\xi,y)=\mu\epsilon_\infty+\int\frac{dk}{2\pi}e^{ik\xi}K_{yy}(k,y)\hat{U}^{(+)}(k) , \tag{3.23}$$

$$\tilde{\sigma}_{xy}(\xi,y)=\int\frac{dk}{2\pi}e^{ik\xi}K_{xy}(k,y)\hat{U}^{(+)}(k) , \tag{3.24}$$

with

$$K_{xx}(k,y)=\frac{2\mu k}{v^2}\left\{\left[1-\left[1-\frac{2}{\kappa}\right]\beta_l^2\right]\left[G(k)\cosh(k\beta_l y)+\frac{1+\beta_t^2}{2\beta_l}\sinh(k\beta_l y)\right]-\frac{2\beta_t}{\kappa}[\sinh(k\beta_t y)+H(k)\cosh(k\beta_t y)]\right\} , \tag{3.25}$$

$$K_{yy}(k,y)=\frac{-2\mu k}{\kappa v^2}\left\{(1+\beta_t^2)\left[G(k)\cosh(k\beta_l y)+\frac{1+\beta_t^2}{2\beta_l}\sinh(k\beta_l y)\right]-2\beta_t[\sinh(k\beta_t y)+H(k)\cosh(k\beta_t y)]\right\} , \tag{3.26}$$

$$K_{xy}(k,y)=\frac{2i\mu k}{\kappa v^2}\{(1+\beta_t^2)[\cosh(k\beta_t y)+H(k)\sinh(k\beta_t y)]-[2\beta_l G(k)\sinh(k\beta_l y)+(1+\beta_t^2)\cosh(k\beta_l y)]\} , \tag{3.27}$$

and we are interested in the limits $\xi\to 0^-$ and $y\to 0$.

The Weiner-Hopf technique is to separate the functions of Eq. (3.16) into two groups, each of which contains singularities only in the upper- or lower-half plane. The two groups, being equal to each other, must each be equal to an entire function which can be taken to be a constant K. To achieve this, first write $\hat{P}(k)$ as the product $\hat{P}^{(+)}(k)\hat{P}^{(-)}(k)$ and rewrite Eq. (3.16) as

$$ik\hat{P}^{(+)}(k)\hat{U}^{(+)}(k)+ik\hat{\Lambda}^{(+)}(k)=\frac{1}{\hat{P}^{(-)}(k)}[\epsilon_\infty-ik\hat{\epsilon}_0^{(-)}(k)]-ik\hat{\Lambda}^{(-)}(k) . \tag{3.28}$$

Here,

$$\hat{\Lambda}^{(\pm)}(k)=\mp\int_{C^{(\mp)}}\frac{dk'}{2\pi i}\frac{1}{(k'-k)}\frac{\hat{\epsilon}_c^{(+)}(k')}{\hat{P}^{(-)}(k')} , \tag{3.29}$$

where the contours $C^{(+)}$ and $C^{(-)}$ go from $-\infty$ to $+\infty$ in the k' plane passing, respectively, above and below the pole at $k'=k$. The constant K is then obtained by evaluating both sides of Eq. (3.28) in the limit $k\to 0$ and using $\lim_{k\to 0}ik\hat{U}^{(+)}(k)=U(\infty)$:

$$K=\hat{P}^{(+)}(0)U(\infty)=\frac{\epsilon_\infty}{\hat{P}^{(-)}(0)} . \tag{3.30}$$

From (3.17) and (3.18),

$$\hat{P}(0)=\hat{P}^{(+)}(0)\hat{P}^{(-)}(0)=1 ;$$

thus, (3.30) assures $U(\infty)=\epsilon_\infty$, as desired. Without loss of generality, we normalize $\hat{P}^{(\pm)}(k)$ so that

$$\hat{P}^{(+)}(0)=\hat{P}^{(-)}(0)=1 .$$

Hence, the formal results are

$$ik\hat{U}^{(+)}=\frac{1}{\hat{P}^{(+)}(k)}[\epsilon_\infty-ik\hat{\Lambda}^{(+)}(k)] \tag{3.31}$$

and

$$ik\hat{\epsilon}_0^{(-)}(k)=\epsilon_\infty[1-\hat{P}^{(-)}(k)]-ik\hat{P}^{(-)}(k)\hat{\Lambda}^{(-)}(k) . \tag{3.32}$$

IV. EXPLICIT RESULTS

To proceed further, one needs explicit representations for the factors $\hat{P}^{(\pm)}(k)$. One could carry out the factorization numerically [12]. On the other hand, we find from (3.18) that

$$\hat{F}(k)=\begin{cases}1+O(k^2) , & |k|\ll 1\\ b(v)|k| , & |k|\gg 1\end{cases} \tag{4.1}$$

with

$$b(v)=\frac{\beta_t}{\kappa v^2}\left[4-\frac{(1+\beta_t^2)^2}{\beta_t\beta_l}\right] , \tag{4.2}$$

which vanishes at $v=c_R$ and is greater than zero for v

less than c_R. We therefore approximate $\hat{F}(k)$ as

$$\hat{F}(k) \sim \sqrt{1+b^2k^2} , \tag{4.3}$$

which produces the right behavior in both the small and large $|k|$ limits and which allows us to carry out the analysis explicitly. In effect, we replace the complex poles and zeros of $\hat{F}(k)$ by a pair of branch points. This approach was first adopted by Barber, Donley, and Langer [13]. To be consistent with this approximation, we also take

$$-kG(k) \cong \frac{1+\beta_t^2}{2\beta_l b}\sqrt{1+b^2k^2} + \frac{\kappa}{2} - \frac{1+\beta_t^2}{2\beta_l b} , \tag{4.4}$$

$$-kH(k) \cong \frac{1}{b}\sqrt{1+b^2k^2} + \frac{\kappa}{2\beta_t} - \frac{1}{b} . \tag{4.5}$$

This approximation is checked to reproduce the Irwin-Williams equilibrium asymptotic stress field [14] when the crack speed v becomes vanishingly small. We therefore believe that this approach does not miss any essential features of the problem.

Our approximate form for $\hat{P}(k)$ is then

$$\hat{P}(k) \cong \sqrt{1+b^2k^2} + i\tilde{\eta}vk^3 , \tag{4.6}$$

which has the same form as the kernel $\hat{Q}(k)$ in (3.3) of LN. The three roots of $\sqrt{1+b^2k^2}+i\tilde{\eta}vk^3=0$ are denoted by k_0, k_1, and k_2. The root $k_0 \equiv -ip_0$ is always on the negative imaginary k axis; k_1 is complex with positive real and imaginary parts, $k_2=-k_1^*$, and $-i\tilde{\eta}vk_0k_1k_2=+1$. The factorization of $\hat{P}(k)$ is then accomplished by using the Cauchy method [9,15] to write

$$\ln\left[\frac{\hat{P}(k)}{i\tilde{\eta}v(k-k_0)(k-k_1)(k-k_2)}\right] = \Phi^{(+)}(k) + \Phi^{(-)}(k) , \tag{4.7}$$

and $\Phi^{(\pm)}$ are expressed as integrals over the discontinuities across the corresponding branch cuts of the branch points at $\pm i/b$:

$$\Phi^{(\pm)}(k) = \mp \int_1^\infty \frac{dp}{\pi} \frac{\theta(p)}{(p \pm ibk)} , \tag{4.8}$$

where

$$\theta(p) = \arctan\left[\frac{\sqrt{p^2-1}}{\gamma p^3}\right] \tag{4.9}$$

and $\gamma = \tilde{\eta}v/b^3$. The explicit forms for $\hat{P}^{(\pm)}$ are

$$\hat{P}^{(+)}(k) = -\tilde{\eta}vp_0(k-k_1)(k-k_2) \times \exp[\Phi^{(+)}(k)-\Phi^{(+)}(0)] , \tag{4.10}$$

$$\hat{P}^{(-)}(k) = \left[1-\frac{ik}{p_0}\right]\exp[\Phi^{(-)}(k)-\Phi^{(-)}(0)] . \tag{4.11}$$

The $\Phi^{(\pm)}(0)$ are included to give the normalization $\hat{P}^{(\pm)}(0)=1$.

We are interested in the stress distribution at the crack tip. We get $\epsilon_0(\xi)$ by inverse Fourier transforming (3.32) and $\tilde{\sigma}_{yy}(\xi \to 0^-,0)$ is just $\tilde{\sigma}_0(0^-)=\mu\tilde{\epsilon}_0(0^-)$. From (3.25) and (3.26) and using (4.4) and (4.5), we have

$$K_{xx}(k,0) = A(v)K_{yy}(k,0) - \mu\left[1-\frac{2}{\kappa}\right] , \tag{4.12}$$

where

$$A(v) \equiv \frac{(1+\beta_t^2)(1+2\beta_l^2-\beta_t^2)-4\beta_t\beta_l}{4\beta_t\beta_l-(1+\beta_t^2)^2} . \tag{4.13}$$

Hence, for $\xi<0$,

$$\tilde{\sigma}_{xx}(\xi,0) = \mu\left[1-\frac{2}{\kappa}\right]\epsilon_\infty + A(v)[\tilde{\sigma}_{yy}(\xi,0)-\mu\epsilon_\infty] . \tag{4.14}$$

From (3.24) and (3.27), we get

$$\tilde{\sigma}_{xy}(\xi,0)=0 . \tag{4.15}$$

With the use of the approximation (4.6) for $\hat{P}(k)$, the present problem has the same mathematical structure as that of LN; thus our results resemble those presented there:

$$\tilde{\epsilon}_0(0^-) = \epsilon_c\{U(0^+)\} + \left[p_0+\frac{p_\infty}{b}\right]\left[\frac{\epsilon_\infty}{p_0}e^{\Phi^{(-)}(0)} - \bar{\epsilon}\int_0^\lambda T(\xi)d\xi\right] - \bar{\epsilon}T(\lambda) , \tag{4.16}$$

$$U(\lambda) \equiv \frac{\delta}{W} = \frac{\bar{\epsilon}}{\tilde{\eta}v}\int_0^\lambda T(x)dx \int_0^x S(z)dz , \tag{4.17}$$

where

$$T(z) = -\int \frac{dk}{2\pi i}\frac{e^{-\Phi^{(-)}(k)-ikz}}{k+ip_0} , \tag{4.18}$$

$$S(z) = -\int \frac{dk}{2\pi}\frac{e^{-\Phi^{(+)}(k)+ikz}}{(k-k_1)(k-k_2)} , \tag{4.19}$$

$$p_\infty = \frac{1}{\pi}\int_1^\infty \theta(p)dp , \tag{4.20}$$

and $\bar{\epsilon}=\bar{\sigma}/\mu$ is the yield strain. We see that $\tilde{\epsilon}_0(0^-)$ is nonsingular, as discussed before. The new Barenblatt condition is obtained [9] by requiring the traction on the crack surface just behind the crack tip to be finite:

$$\frac{\epsilon_\infty}{\bar{\epsilon}} = p_0 e^{\Phi^{(-)}(0)}\int_0^\lambda T(\xi)d\xi , \tag{4.21}$$

which also ensures the continuity of stress at the crack tip. Thus, we have

$$\tilde{\sigma}_{yy}(0^-,0) = \mu\tilde{\epsilon}_0(0^-) = \bar{\sigma}[1-T(\lambda)] . \tag{4.22}$$

The other stress components are given by (4.14) and (4.15). For $v \to 0$, $A(v)=1$; therefore,

$$\tilde{\sigma}_{xx}(0^-,0) = -\frac{2}{\kappa}\mu\epsilon_\infty + \tilde{\sigma}_{yy}(0^-,0) < \tilde{\sigma}_{yy}(0^-,0) , \tag{4.23}$$

as one would expect for crack propagating along the x-direction. However, for larger speeds, as $A(v)$ is a monotonic increasing function of v, $\tilde{\sigma}_{xx}(0^-,0)$ may overtake $\tilde{\sigma}_{yy}(0^-,0)$. If this occurs, crack growth along the x

direction will not release the maximum stress.

To investigate this, define

$$\sigma_d \equiv \tilde{\sigma}_{xx}(0^-,0)-\tilde{\sigma}_{yy}(0^-,0) \quad (4.24)$$

and study its behavior as a function of the crack speed v. Using (4.14) and (4.22), we get

$$R_d \equiv \frac{\sigma_d}{\bar{\sigma}} = [A(v)-1][1-T(\lambda)] - \left[A(v)-1+\frac{2}{\kappa}\right]\frac{\epsilon_\infty}{\bar{\epsilon}} . \quad (4.25)$$

The most interesting behavior of the model [9,10] occurs when the system parameters are such that there exists a regime where $\lambda/b \ll \sqrt{\gamma} \ll 1$. In this regime, we have the following approximations:

$$T(z) \approx 1-\frac{z}{2b\sqrt{\gamma}} , \quad z \ll \sqrt{\gamma}$$
$$S(z) \approx z , \quad z \ll \sqrt{\gamma} . \quad (4.26)$$

Using (4.26) in (4.17) and (4.21), we get the following results which are the same as those obtained in LN:

$$\gamma \approx \frac{4}{81}\left[\frac{\delta}{W\bar{\epsilon}}\right]^2 \left[\frac{\epsilon_\infty}{\epsilon_G}\right]^{12} , \quad (4.27)$$

$$\frac{\lambda}{b} \approx \frac{2}{3}\left[\frac{\delta}{W\bar{\epsilon}}\right]\left[\frac{\epsilon_\infty}{\epsilon_G}\right]^4 , \quad (4.28)$$

where

$$\epsilon_G \equiv \epsilon_\infty(v\to 0) = \left[\frac{2\bar{\epsilon}\delta}{W}\right]^{1/2} \quad (4.29)$$

is the Griffith threshold at which crack propagation first becomes energetically possible. Hence we have

$$\frac{\sigma_d}{\bar{\sigma}} \approx \left[\frac{\delta}{W\bar{\epsilon}}\right]^{1/3} \left\{ \frac{3}{2}[A(v)-1]\left[\frac{4}{81\gamma}\right]^{1/6} - \sqrt{2}\left[A(v)-1+\frac{2}{\kappa}\right]\left[\frac{81\gamma}{4}\right]^{1/12} \right\} . \quad (4.30)$$

The range of validity of (4.30) is determined by the self-consistency of (4.27) and (4.28), with $\lambda/b \ll \sqrt{\gamma} \ll 1$, and is given by

$$36\left[\frac{\delta}{W\bar{\epsilon}}\right]^2 \ll \gamma \ll 1 . \quad (4.31)$$

Thus, (4.30) has a large range of validity whenever the macroscopic length W is very large. The corresponding range of velocities is determined by the value of $\tilde{\eta}$. In the following section, (4.30) will be evaluated numerically and the variation of R_d with the crack speed v will be studied.

V. INTERPRETATION AND DISCUSSION

The quantity R_d depends only on v with κ, $\tilde{\eta}$, and $\delta/(W\bar{\epsilon})$ being system-dependent parameters. In practice [11], Possion's ratio ν varies from 0 to 1/2, so we have [see Eq. (2.4)]

$$\kappa \geq 2 \quad \text{(plane strain)}$$
$$4 \geq \kappa \geq 2 \quad \text{(plane stress)} . \quad (5.1)$$

The system-dependent parameter

$$\frac{\delta}{W\bar{\epsilon}} = \frac{1}{2}\left[\frac{\epsilon_G}{\bar{\epsilon}}\right]^2 \quad (5.2)$$

must be very small in any realistic situation so that the yield strain $\bar{\epsilon}$ greatly exceeds the Griffith threshold ϵ_G. In the numerical calculation, κ and $\delta/(W\bar{\epsilon})$ are fixed to be 3 and 10^{-10}, respectively. The value of γ allowed is between 2.3×10^{-11} and 10^{-4} so that $\lambda/b \leq 1/20\sqrt{\gamma}$ and $\sqrt{\gamma} \leq 0.01$. The corresponding value of $\epsilon_\infty/\epsilon_G$ ranges between 7 and 28. The values of $\tilde{\eta}$ are chosen so that a reasonably large range of velocities is covered under the above constraints on γ. The results are displayed in Fig. 2.

It is reasonable to expect a crack to propagate perpendicular to the direction of the maximum principal stress. Thus, one expects R_d to be negative. This is indeed the case when the crack speeds are small for all the values of $\tilde{\eta}$ studied. However, when the crack moves faster than some critical value v_c, R_d becomes positive, indicating that steady-state crack propagation at such speeds is probably unstable. In fact, if the crack continues to propagate perpendicular to the direction of the maximum principal stress, it would have to propagate off the x direction. We note that Marder and Liu [16] recently found that fracture of a triangular lattice involves breaking of bonds off the crack path when its speed is greater than 0.67 of the sound speed.

Furthermore, consider the two planes (or lines in our two-dimensional case) which make a small angle τ with the negative x axis. The shear stress acting at the crack

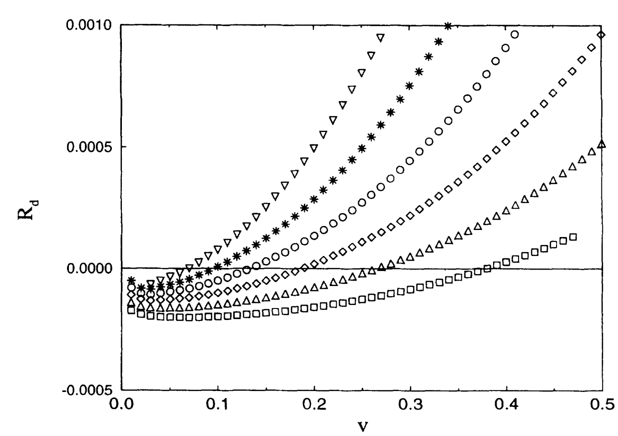

FIG. 2. R_d as a function of crack speed v for different values of $\tilde{\eta}$. The speed v is measured in units of the transverse wave speed c_t of the material. The values of $\tilde{\eta}$ are 10^{-10} (downward triangles), 10^{-9} (stars), 10^{-8} (circles), 10^{-7} (diamonds), 10^{-6} (upward triangles), and 10^{-5} (squares).

tip, along these planes, is given by

$$\sigma_s = [\sigma_{xx}(0^-,0) - \sigma_{yy}(0^-,0)]\sin\tau\cos\tau = \sigma_d \sin\tau\cos\tau \ . \quad (5.3)$$

At small crack speeds, $\sigma_d < 0$ so σ_s is negative, which opposes any tendency for the crack to deviate from its original direction. However, for speeds greater than v_c, σ_s is positive. Suppose the crack starts to deviate slightly from its original direction; this positive shear stress will then act in a direction that enhances the deviation, again indicating that steady-state propagation will be unstable.

From (4.30), note that the critical speed v_c depends only on $\tilde{\eta}$. Recall that $\tilde{\eta} = \eta c_t / (\mu W^2)$, so v_c depends on both material properties and the width of the strip. The dependence of v_c on $\tilde{\eta}$ is shown in Fig. 3. We should mention that this result does not quite agree with the experimental finding of Fineberg *et al.* [3] that the critical velocity is independent of sample geometry. On the other hand, values of v_c close to those observed experimentally can be obtained when $\tilde{\eta}$ is chosen appropriately.

Hence, we have shown that the crack tip stress field changes qualitatively when the speed of propagation is larger than some critical value. Such a dynamical modification has the interesting indication that steady-state crack propagation above the critical speed is probably unstable.

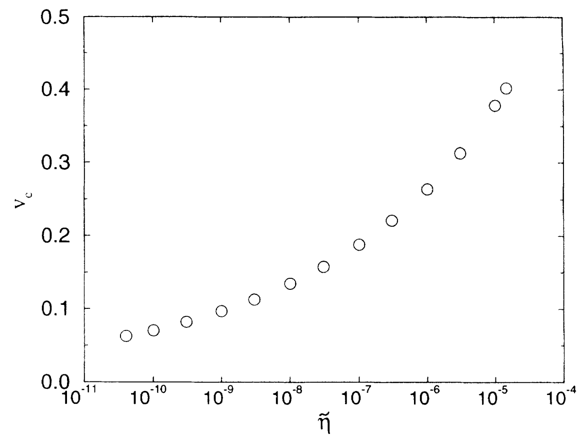

FIG. 3. The dependence of the critical speed v_c on the parameter $\tilde{\eta}$ with v_c measured in units of the transverse wave speed c_t of the material.

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[1] Useful general references are M. F. Kanninen and C. H. Popelar, *Advanced Fracture Mechanics* (Oxford University Press, New York, 1985); L. B. Freund, *Dynamic Fracture Mechanics* (Cambridge University Press, New York, 1990).
[2] K. Ravi-Chandar and W. G. Knauss, Int. J. Fract. **26**, 141 (1984).
[3] J. Fineberg, S. P. Gross, M. Marder, and H. L. Swinney, Phys. Rev. Lett. **67**, 457 (1991); Phys. Rev. B **45**, 5146 (1992).
[4] E. H. Yoffe, Philos. Mag. **42**, 739 (1951).
[5] J. W. Craggs, J. Mech. Phys. Solids **8**, 66 (1960).
[6] B. R. Baker, J. Appl. Mech. **29**, 449 (1962).
[7] G. I. Barenblatt, Adv. Appl. Mech. **7**, 56 (1962). See also, Ref. [1] for various discussions of the cohesive zone.
[8] J. S. Langer, Phys. Rev. A **46**, 3123 (1992).
[9] J. S. Langer and H. Nakanishi, Phys. Rev. E **48**, 439 (1993).
[10] J. S. Langer, Phys. Rev. Lett. **70**, 3592 (1993).
[11] See, for example, L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, 3rd ed. (Pergamon, New York, 1986).
[12] W. G. Knauss, J. Appl. Mech. **33**, 356 (1966).
[13] M. Barber, J. Donley, and J. S. Langer, Phys. Rev. A **40**, 366 (1989).
[14] G. R. Irwin, J. Appl. Mech. **24**, 361 (1957); M. L. Williams, *ibid.* **24**, 109 (1957).
[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Part I.
[16] M. Marder and Xiangming Liu, Phys. Rev. Lett. **71**, 2417 (1993).